MATH.MA.210 Discrete Mathematics Exam, 22.09.2026

You are to answer five out of the following six questions. Do not submit more answers; only the first five will be graded. If you have requested a special arrangement, only answer four and mark the front page with "Special Arrangement" text.

(All questions have equal value.)

Questions

1. The universal set in this question is the set of nonnegative integers $\mathbb{Z}^+ \cup \{0\}$ (or natural numbers with 0, if you will). You are asked to define sets or write logical predicates. You are allowed to use arithmetic operations such as multiplication and addition in your answers and assume their properties are what one would expect.

   (a) Define the set E of integers that are divisible by 4.

   (b) Write a predicate $p(x)$ that states that x is a product of three (not necessarily distinct) primes.

   (c) Define the set S consisting of integers that can be written as a sum of two squares.

2. Prove the following statement for all $n \in \mathbb{Z}^+$ using induction:

$$\sum_{i=1}^{n} 2i = n(n+1).$$

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A by

$$(a, b) \in R \iff \gcd(a, b) > 1.$$

   (a) Is R reflexive? symmetric? transitive? Justify your answers.

   (b) Is R an equivalence relation? Explain.

   (c) Do we get an equivalence relation if we modify the relation to

$$a \sim b \iff \gcd(a, b) = 1?$$

   If so, what are the equivalence classes? If not, show a counterexample.

4. A student is choosing a small study package. There are three sets of courses available:

$$M = \{m_1, m_2, m_3, m_4, m_5\}, \quad P = \{p_1, p_2, p_3, p_4\}, \quad C = \{c_1, c_2, c_3, c_4\}.$$

   The student chooses exactly 6 courses in total.

   (a) How many ways are there to choose the 6 courses with exactly 2 courses from each of M, P, and C?

(b) How many ways are there to choose the 6 courses with exactly 2 courses from M, and the remaining courses freely from P and C?

(c) How many ways are there to choose the 6 courses if the student must choose either at least 2 courses from M or at least 2 courses from P (or both)?

5. Let D_3 be the dihedral group of symmetries of an equilateral triangle; that is, symmetries that preserve the triangle.

   (a) List all elements of D_3 and describe them geometrically.

   (b) Show that D_3 forms a group under composition of symmetries.

   (c) Is D_3 commutative? Justify your answer by giving a counterexample if it is not.

6. (a) What is the remainder when 2^{2027} is divided by 13?

   (b) Solve the congruence

   $$9x \equiv 4 \pmod{11}.$$

   (c) Determine whether the equation

   $$8x \equiv 3 \pmod{14}$$

   has a solution. Justify your answer.

Appendix: Definitions

Definition 1. A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

Definition 2. A relation is a partial order if it is reflexive, antisymmetric, and transitive.

Definition 3. A group $(G, \cdot)$ consists of a set G with an operation $\cdot$ such that:

1. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Identity: $\exists e \in G$ such that $e \cdot a = a \cdot e = a$
3. Inverse: $\forall a \in G, \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition 4. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we say $a \equiv b \pmod{n}$ if $n \mid (a - b)$.

Definition 5. For $n, k \in \mathbb{Z}^+$ with $0 \leq k \leq n$, the binomial coefficient ("n choose k") is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$