MATH.MA.210 Discrete Mathematics Exam, 16.06. 2026

You are to answer five out of the following six questions. Do not submit more answers; only the first five will be graded. If you have requested a special arrangement, only answer four and mark the front page with "Special Arrangement" text.

(All questions have equal value.)

Questions

1. The universal set in this question is the set of nonnegative integers $\mathbb{Z}^+ \cup \{0\}$ (or Natural numbers with 0, if you will). You are asked to define sets or write logical predicates. You are allowed to use arithmetic operations such as multiplication and addition in your answers and assume their properties are what one would expect.

   (a) Define the set F of integers that are divisible by 4.

   (b) Write a predicate $p(x)$ that states that x is a prime number.

   (c) Using F and $p(x)$ above, write the following claim "Every number that is divisible by four can be expressed as a sum of three primes". (The claim is false, but write the formula)

2. We define the following: Let

   $$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

   be the n-th harmonic number. Prove the following statement by induction for all $n \in \mathbb{Z}^+$:

   $$\sum_{i=1}^{n} H_i = (n+1)H_n - n$$

   Make sure you follow the induction proof scheme.

3. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define a relation R on A by

   $$(a, b) \in R \iff a | b$$

   (a) Is R reflexive? symmetric? transitive? Justify your answers.

   (b) Is R a partial order?

   (c) Define an equivalence relation $\sim$ that is not the same as equality, and for which there exist two elements $a, b \in A$ such that $a \not\sim b$.

4. A student is choosing a small study package. There are three sets of courses available:

   $$M = \{m_1, m_2, m_3, m_4\}, \quad P = \{p_1, p_2, p_3, p_4\}, \quad C = \{c_1, c_2, c_3, c_4\}.$$

   The student chooses exactly 5 courses in total.

(a) How many ways are there to choose the 5 courses with exactly 2 courses from M, exactly 2 from P, and exactly one course from C?

(b) How many ways are there to choose the 5 courses with at least two courses from M, and the rest freely?

(c) How many ways are there to choose the 5 courses so that there is at least one course from each set?

5. Let D_3 be the dihedral group of symmetries of an equilateral triangle; that is, symmetries that preserve the shape.

   (a) List all elements of D_3 and describe them geometrically.

   (b) Show that D_3 forms a group under composition of symmetries.

   (c) Is D_3 commutative? Justify your answer by giving a counterexample if it is not.

6. (a) What is the remainder when 3^{2026} is divided by 7?

   (b) Solve the congruence

$$5x \equiv 4 \pmod{9}.$$

   (c) Determine whether the equation

$$8x \equiv 6 \pmod{14}$$

   has a solution. Justify your answer.

Appendix: Definitions

Definition 1. A relation is an equivalence relation if it is reflexive, symmetric, and transitive.
Definition 2. A relation is a partial order if it is reflexive, antisymmetric, and transitive.
Definition 3. A group $(G, \cdot)$ consists of a set G with an operation $\cdot$ such that:

1. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
2. Identity: $\exists e \in G$ such that $e \cdot a = a \cdot e = a$
3. Inverse: $\forall a \in G, \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition 4. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we say $a \equiv b \pmod{n}$ if $n \mid (a - b)$.
Definition 5. For $n, k \in \mathbb{Z}^+$ with $0 \leq k \leq n$, the binomial coefficient ("n choose k") is defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$